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COMMENT

Scale-dependent lacunarity and fractal Ising models

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Abstract. As an alternate way of measuring the lacunarity of Sierpinski carpets, we have computed the site to site fluctuation of the quantity $M(n)$, namely the total number of lattice sites within the chemical distance of n. In long scales of n, the resultant lacunarity agrees qualitatively with the results obtained in the past using the covering method. In short scales of n, however, the lacunarity shows a different trend among fractals; it varies monotonically with the Ising correlation length exponent, a trend which was sought before but not found with the previous method. This applies only for the carpets of the first kind.

The phase transitions in hypercubic lattices show a remarkable degree of universality (for a picturesque exposition of the universality idea see [1]), but the situation in fractal lattices is much more complex. According to the pioneering work of Gefen *et al* [2-6], the critical properties depend not only on the fractal dimensionality but also on other geometrical factors such as the order of ramification, the connectivity and the lacunarity [7]. Ensuing works [8-10] tend to indicate that the problem is even more complex than originally thought. In particular, the relevance of lacunarity is controversial. The purpose of this comment is to present the results of a lacunarity calculation which has been computed in a different way than in the past. The results show a new feature which seems to offer a small, but nevertheless encouraging, hope for the possibility of universality in the fractal critical behaviour.

The purpose of lacunarity is to measure the extent of the failure of fractals to be translationally invariant. Gefen *et al* [6] examine the critical behaviour of Ising spins placed on the carpets shown in figure 1. Both carpets have the same amount of total cut-out area and thus the same fractal dimensionality, but the total cut-out has been

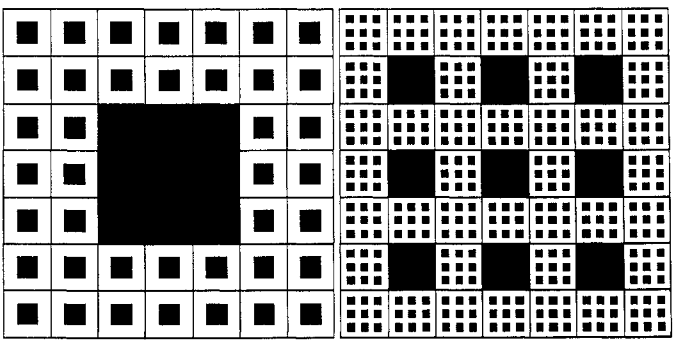

Figure 1. Sierpinski carpets studied in [6].

divided into a different number of holes such that the lacunarity may be varied while keeping the connectivity fixed. According to their analysis, which is based on the Migdal–Kadanoff bond moving renormalisation group transformation, the correlation length exponent increases with decreasing lacunarity. It is this conjecture which is controversial.

The lacunarity is estimated in the above study from a simple covering method. A square window of a size equal to the total cut-out is placed at various positions over the fractals and then the number of non-eliminated basic subsquares underneath the window is counted. The site-to-site fluctuation of this number is taken as the approximate lacunarity. This provides a convenient method, but it fails to ensure that zero lacunarity is a necessary and sufficient condition for translational invariance [9, 11]. This is due to the fact that one particular length scale has been singled out for no compelling reason by using only one window of a fixed size. Thus Lin and Yang [11] propose a multi-window scheme in which the window size varies from the total cut-out area to a properly chosen larger value. The results for different sizes are averaged for the final value of lacunarity. But Wu and Hu [9] argue that there is no compelling reason to exclude the short length scales, and revise the method so that it may work correctly for a number of thoughtfully chosen group of carpets. By 'correctly', it is meant that the lacunarity reflects the extent of deviation from translational invariance, which is largely a visual impression.

Using the lacunarity computed in this way, Wu and Hu [9] re-examine the role of lacunarity in determining the critical behaviour. They study the Ising critical behaviour in the six carpets shown in figure 2. These carpets are similar to those in figure 1, but the way the total cut-out is divided into various numbers of holes, and the way the holes are moved around, are more complex. The fractal dimensionality and connectivity are again the same in all six carpets, but the lacunarity decreases monotonically as we go from (a) to (f). Wu and Hu show, however, that the correlation length exponent ν does not increase monotonically with decreasing lacunarity as expected from Gefen *et al.* As the first two columns of table 1 show, the variation of the exponent has no predictable relationship with that of the lacunarity. The correlation length exponent does not depend on the lacunarity. Instead it depends on the number of holes and on whether the carpet is of the first kind or the second kind (to be explained later). Thus the relevance of lacunarity appears doubtful.

However, this may be due to the particular way the lacunarity has been computed. The mass under the square window is only one of many quantities which fluctuates from site to site. Each of them reflects in a different way the failure of fractals to be translationally invariant. In fact, there is no *a priori* reason why all of them should be relevant to the critical behaviour. In order to be relevant, we believe that the quantity should emphasise the fact that all lattice sites do not provide the same environment to the spins in fractal lattices. Indeed, all spins do not have the same number of nearest neighbours to couple with; the number fluctuates from site to site. This is also true in diluted magnets [12], but whereas the fluctuation is random in diluted magnets, it maintains a definite pattern in deterministic fractal lattices. The same holds true for the next-nearest neighbours, and the next-next-nearest neighbours, etc, but the pattern of the fluctuation could be different in each order. In fact, that turns out to be the case.

Because of the holes, determination of whether a neighbour is next nearest or next-next nearest, etc, should not be judged by the Pythagorean distance. The chemical distance (or the cow distance) [13] would be a much better quantity to work with.

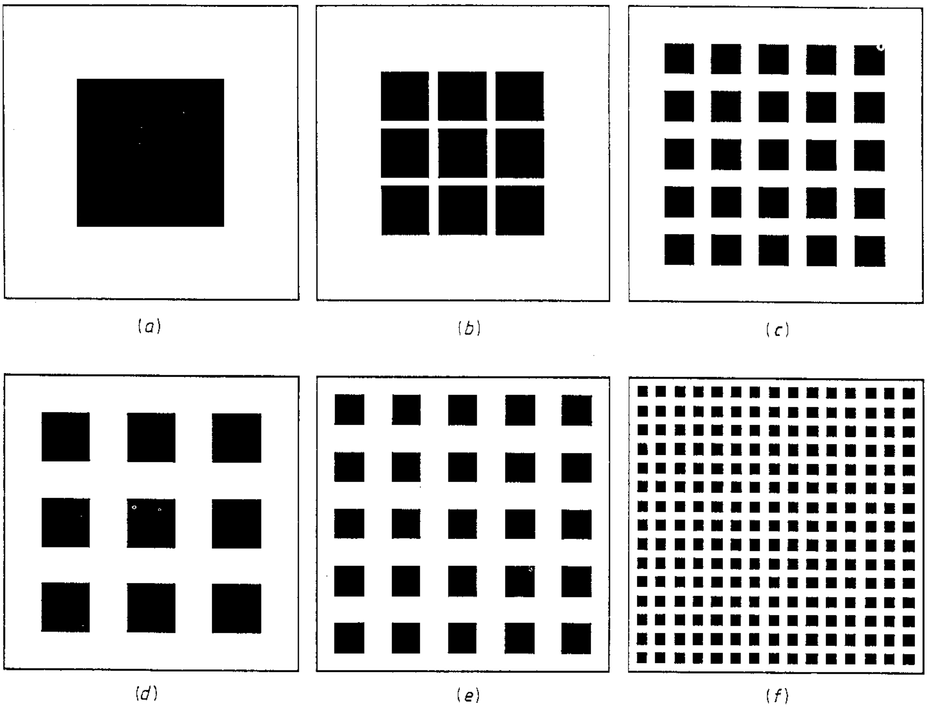

Figure 2. Sierpinski carpets studied in [9]. From a $b \times b$ square, cut out a total of $l \times l$ squares. Do this by making $m \times m$ disconnected holes, each of which consists of $n \times n$ squares. Adjacent holes are searated by p, and there is a distance of q from the edge of the fractal to the nearest holes. For all six fractals, $b = 31$, $l = 15$. (*a*) $m = 1$, $n = 15$, $p = 0$, $q = 8$; (*b*) $m = 3$, $n = 5$, $p = 1$, $q = 7$; (*c*) $m = 5$, $n = 3$, $p = 2$, $q = 4$; (*d*) $m = 3$, $n = 5$, $p = 4$, $q = 4$; (*e*) $m = 5$, $n = 3$, $p = 3$, $q = 2$; (*f*) $m = 15$, $n = 1$, $p = 1$, $q = 1$.

Table 1. L3 is an estimate of the lacunarity for the fractals in figure 2 based on the square covering method. ν is the correlation length exponent (both from [9]). The last two columns represent our results for the lacunarity with $n = 1$, and $n = 32$, respectively.

Case	L3	ν	L(1)	L(32)
(*a*)	0.4217	1.7900	0.5551	0.5642
(*b*)	0.3343	1.8671	0.5593	0.5620
(*c*)	0.1384	1.9531	0.5630	0.5557
(*d*)	0.1248	1.8671	0.5593	0.5551
(*e*)	0.0540	1.9531	0.5630	0.5532
(*f*)	0.0274	1.8539	0.5713	0.5530

Thus we compute for each lattice site the quantity $M(n)$ which represents the number of lattice sites within the chemical distance of n. As a measure of lacunarity, we then compute the site-to-site fluctuation of $M(n)$, i.e.

$$L(n) = (\langle M^2(n)\rangle - \langle M(n)\rangle^2)^{1/2} / \langle M(n)\rangle$$

where the average is taken over all lattice sites.

We digitise the fractals in figure 2 on a 62×62 square lattice and count $M(n)$ using a computer program. A periodic boundary condition is used. Figure 3 shows the result. The lacunarity varies with the detailed construction of fractals as before, but the variation is so dependent on the scale of n that it is meaningless to speak of the lacunarity without a specific reference to the scale of n. For example, carpet (a) is more lacunar than carpet (f) in long scales of n, but the comparison is opposite in short scales. To emphasise this scale dependence, table 1 presents the results of $n = 1$ and $n = 32$ in the fourth and fifth column, respectively. In long scales, the lacunarity decreases as we go from (a) to (f), and thus agrees qualitatively with that of Wu and Hu. In short scales, however, the lacunarity shows an entirely different trend among the fractals. Notice how the short-scale lacunarity varies as we go from (a) to (f), and compare it with the way the exponent ν varies. Excluding carpet (f) for a moment, it is clear that the behaviour of the correlation length exponent is no longer unpredictable as before: it varies monotonically with the short-scale lacunarity. In particular, when two fractals have the same short-scale lacunarity, they also have the same exponent. Notice, however, that this is not the same monotonic relationship expected from Gefen *et al.* The correlation length exponent increases monotonically with increasing (not decreasing) short-scale lacunarity.

Carpet (f) does not follow this trend. In fact, there is a geometrical feature which is present in this carpet but not in the other carpets. Only in carpet (f) are there rows and columns which consist entirely of bonds bordering the cut-out area. Wu and Hu call such carpets the second kind, and the rest the first kind. Thus, we may state that the short-scale lacunarity is a relevant geometric variable for the critical behaviour in the carpets of the first kind.

It is easy to see that the number of holes, which plays an important role in the renormalisation group analysis of Wu and Hu, and the short-scale lacunarity, which we have emphasised above, are actually related to each other. If there are more holes, it means that more sites border the boundaries of the holes. Since these sites have

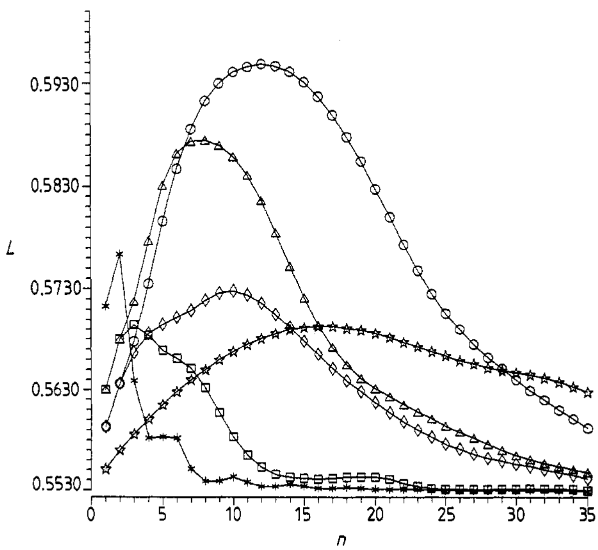

Figure 3. The scale-dependent lacunarity of the carpets shown in figure 2. Because of the periodic boundary conditions, $n = 32$ is the maximum possible distance. (a) ☆, (b) ○, (c) △, (d) ◇, (e) □, (f) $*$.

fewer nearest neighbours than those in the interior of hole-free regions, the short-scale lacunarity tends to become larger. It is not easy, however, to understand physically why the short-scale lacunarity has to be singled out in the carpets of the first kind. To answer this and other questions, it is necessary to understand how spins correlate in fractal lattices. Unfortunately the spin correlation function is hardly understood except in the Koch curves and the Sierpinski gaskets [14]. An extensive study is underway, and we hope that we will be able to understand the carpets of second kind in the future.

We have also computed $M(r)$, namely the total number of lattice sites within the Pythagorean distance of r. It requires a greater amount of computer time, but the resultant lacunarity shows a similar qualitative trend. We have also studied the scaling behaviour of the digitised carpets. The function $M(n)$ shows power-law behaviour with an average chemical dimension [13] of 1.85. The function $M(r)$ also shows power-behaviour with an average fractal dimension of 1.94, which is 1% above the correct value of 1.9223. This is gratifying because the digitised fractals are only first-generation fractals; extending it to the second generation is beyond the limit of supercomputer speed. The minimum dimension which characterises the scaling law for n and r is near unity, but its exact value is hard to determine.

Note added in proof. The authors regret not having been aware of the following references while writing this article: Taguchi Y 1987 *J. Phys. A: Math. Gen.* **20** 6611; Wu Y-K 1988 *J. Phys. A: Math. Gen.* **21** 4251.

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